# A new representation and a unique code of a simple undirected graph

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**Abstract**

In this note we introduce a representation of simple undirected graphs in terms of polynomials and obtain a unique code for a simple undirected graph.

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Let $M$ be the set of all positive integers greater than 1 and which are not primes. Let $n \in M$. Let $V(n)$ be the set of all proper divisors (not equal to 1 or $n$) of $n$. Define a simple undirected graph $G(n) = (V, E)$ with the vertex set $V = V(n)$ and any two distinct vertices $a, b \in V$ are adjacent if and only if $\gcd(a, b) > 1$. From an observation in [1] it follows that any simple undirected graph is isomorphic to an induced subgraph of $G(n)$ for some $n \in M$. For the sake of completeness and further use of the construction we provide a sketch of the proof below.

**Theorem 1.** *Let $G$ be a simple undirected graph. Then $G$ is isomorphic to an induced subgraph of $G(n)$ for some $n \in M$.*

*Proof.* Let $G = (V, E)$ be a simple undirected graph. Let $\{C_1, C_2, \dots, C_k\}$ be the set of all maximal cliques of $G$. For $i = 1, 2, \dots, k$, let $p_i$ be the $i^{\text{th}}$ prime. For each $v \in V$, define $s_1(v) = \prod \{p_j \mid v \in C_j\}$. Now in order to make the values of $s_1(v)$ distinct for distinct vertices we modify $s_1(v)$ by using different powers of primes $p_j$, if required. For each $v \in V$, let $s(v)$ be the modified value of $s_1(v)$. Let $n$ be a multiple of the least common multiple of $\{s(v) \mid v \in V\}$ such that $n \neq s(v)$ for any $v \in V$. Now it is clear that for any $u, v \in V$,

$$\begin{aligned} & u \text{ is adjacent to } v \text{ in } G \iff u, v \in C_i \text{ for some } i \in \{1, 2, \dots, k\} \\ \iff\ & p_i \text{ is a factor of both } s(u) \text{ and } s(v) \text{ for some } i \in \{1, 2, \dots, k\} \\ \iff\ & \gcd(s(u), s(v)) > 1 \iff u \text{ is adjacent to } v \text{ in } G(n). \end{aligned}$$

Thus $G$ is isomorphic to the subgraph of $G(n)$ induced by the set $\{s(v) \mid v \in V\}$ of vertices of $G(n)$. $\square$

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**Example 2.** Consider the graph $G$ in Figure 1. The maximal cliques of $G$ are $C_1 = \{v_1, v_2, v_3, v_4\}$, $C_2 = \{v_2, v_3, v_4, v_5, v_6\}$, $C_3 = \{v_7, v_8, v_{10}\}$, $C_4 = \{v_9, v_{10}\}$ and $C_5 = \{v_{10}, v_{11}\}$. Then

| $v$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $s_1(v)$ | 2 | $2 \cdot 3$ | $2 \cdot 3$ | $2 \cdot 3$ | 3 | 3 | 5 | 5 | 7 | $5 \cdot 7 \cdot 11$ | 11 |
| $s_1(v)$ | 2 | 6 | 6 | 6 | 3 | 3 | 5 | 5 | 7 | 385 | 11 |
| $s(v)$ | 2 | $2 \cdot 3$ | $2^2 \cdot 3$ | $2 \cdot 3^2$ | 3 | $3^2$ | 5 | $5^2$ | 7 | 385 | 11 |
| $s(v)$ | 2 | 6 | 12 | 18 | 3 | 9 | 5 | 25 | 7 | 385 | 11 |

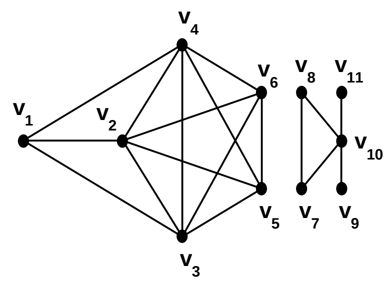


Figure 1: The graph $G$

So $G$ is isomorphic to the subgraph of $G(n)$ induced by the set $\{2, 3, 5, 6, 7, 9, 11, 12, 18, 25, 385\}$ of vertices of $G(n)$, where $n = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 = 69300$.

It is important to note that instead of taking all maximal cliques of $G$ in Theorem 1, it is sufficient to consider a set of cliques of $G$ which covers both vertices and edges of $G$. Now it is clear that if $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ and $m = q_1^{r_1} q_2^{r_2} \cdots q_k^{r_k}$ (where $p_i$'s and $q_j$'s are two sets of distinct primes), then $G(n) \cong G(m)$. Also $G(n)$ is an induced subgraph of $G(nk)$ for any positive integer $k$. Thus for a given simple undirected graph $G$, there are infinite natural numbers $n$ such that $G$ is an induced subgraph of $G(n)$. In order to choose a unique one, we choose the minimum.

**Definition 3.** Let $G$ be a simple undirected graph with $m$ vertices. Let $n$ be the least positive integer such that $G$ is isomorphic to an induced subgraph of $G(n)$. The graph $G(n)$ is called the *mother graph* of $G$. Suppose $G$ is induced by the set $\{n_1, n_2, \ldots, n_m\}$ of vertices of $G(n)$. We make each integer $n_i$ square-free and arrange them according to the non-decreasing order. In this way we get a finite non-decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of square-free positive integers ($> 1$) of length $m$. Such a sequence is called a *coding* of the graph $G$. Now the number of induced subgraphs of $G(n)$, which are isomorphic to $G$, is finite. So the number of codings of $G$ is also finite. We arrange all such codings in the lexicographic ordering considering them as elements in $\mathbb{R}^m$. The least element (coding) in this ordering is called the *code* of $G$ and is denoted by $C(G)$.

For example, $(2, 3, 3, 5, 5, 6, 6, 6, 7, 11, 385)$ and $(2, 3, 3, 5, 5, 7, 10, 10, 10, 11, 231)$ are two codings of the graph $G$ in Example 2 whereas the code of $G$ is $(2, 2, 3, 3, 5, 7, 10, 10, 10, 11, 231)$. Note that the mother graph is unique for a given simple undirected graph $G$ and by definition $C(G)$ is also unique for $G$. Thus we have the following result:

**Theorem 4.** *The mother graph of a simple undirected graph is unique. Let $G_1$ and $G_2$ be two simple undirected graphs. Then $G_1 \cong G_2$ if and only if $C(G_1) = C(G_2)$.*

**Remark 5.** An interesting fact is that the mother graph of $G(n)$ may not be $G(n)$ itself. For example, the mother graph of $G(45)$ is $G(12)$ and $C(G(45)) = C(G(12)) = (2, 2, 3, 6)$. Note that $G(45) \cong G(12) \cong G(p^2q)$ for any two distinct primes $p, q$. In general, it is easy to show that if $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \in M$, where $p_i$'s are distinct primes and $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_k \geqslant 1$, then the mother graph of $G(n)$ is $G(m)$, where $m = t_1^{r_1} t_2^{r_2} \cdots t_k^{r_k}$, such that $t_i$ is the $i^{\text{th}}$ prime, $i = 1, 2, \ldots, k$. Further the coding of $G(n)$ is unique as $G(n)$ is isomorphic to only one induced subgraph of $G(m)$, namely, $G(m)$ itself.

**Definition 6.** Let $G = (V, E)$ be a simple undirected graph. Let $S = \{C_1, C_2, \ldots, C_k\}$ be a set of cliques of $G$ such that $S$ covers both vertices and edges of $G$. Let $m(v) = \prod \{x_j \mid v \in C_j\}$ be a monomial in the polynomial semiring $\mathbb{Z}_0^+[x_1, x_2, \ldots, x_k]$ of $k$ variables over the semiring of non-negative integers with usual addition and multiplication. Define

$$f(G) = f(x_1, x_2, \ldots, x_k) = \sum_{v \in V} m(v).$$

Then $f(G)$ is said to be a *polynomial representation* of $G$.

Consider the graph $G$ in Example 2. Then $f(G) = x_1 + 2x_2 + 2x_3 + x_4 + x_5 + 3x_1x_2 + x_3x_4x_5$ is a polynomial representation of $G$. Now let $G = (V, E)$ be a simple undirected graph and $C = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a coding of $G$. Let $V = \{v_1, v_2, \ldots, v_m\}$ such that the vertex $v_i$ corresponds to $\lambda_i$ for each $i = 1, 2, \ldots, m$. We write $\lambda_i = \mu(v_i)$, $i = 1, 2, \ldots, m$. Let $P = \{p_1, p_2, \ldots, p_k\}$ be the set of all primes $p$ such that $p$ divides $\lambda_i$ for some $i = 1, 2, \ldots, m$. Now for each $j = 1, 2, \ldots, k$, define $S_j = \{v_i \in V \mid p_j \text{ divides } \mu(v_i) = \lambda_i\}$. Then it is easy to see that $S = \{S_j \mid j = 1, 2, \ldots, k\}$ is a set of cliques of $G$ which covers both vertices and edges of $G$. Moreover $v \in S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_r}$ and $v \notin S_j$ for all $j \in \{1, 2, \ldots, k\} \smallsetminus \{i_1, i_2, \ldots, i_r\}$ if and only if $p_{i_1} p_{i_2} \cdots p_{i_r}$ divides $\mu(v)$ but $p_j$ does not divide $\mu(v)$ for all $j \in \{1, 2, \ldots, k\} \smallsetminus \{i_1, i_2, \ldots, i_r\}$. Now since $C$ is a coding, $\mu(v)$ is a product of some distinct primes in $P$ and hence $\mu(v) = p_{i_1} p_{i_2} \cdots p_{i_r}$. Let $f(G)$ be the polynomial representation of $G$ with respect to $S$. Then from the above construction of $S$ and by Definition 6 it follows that $f(G)$ can also be obtained from $C$ by replacing primes $p_i$ by $x_i$ ($i < j \Leftrightarrow p_i < p_j$) and commas by the addition symbol. Thus corresponding to every coding of $G$ there is a polynomial representation of $G$.

**Definition 7.** Let $G$ be a simple undirected graph and $C(G)$ be the code of $G$. Then the polynomial representation of $G$ corresponding to $C(G)$ is called the *normal polynomial representation* or the *canonical polynomial representation* of $G$ and is denoted by $F(G)$.

The normal polynomial representation of the graph $G$ in Example 2 is given by $F(G) = 2x_1 + 2x_2 + x_3 + x_4 + x_5 + 3x_1x_3 + x_2x_4x_5$.

It is also interesting to have a formula for $F(G(n))$.

**Theorem 8.** *Let $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \in M$, where $p_i$'s are distinct primes and $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_k \geqslant 1$. Then $F(G(n))$ contains all the monomials $x_{i_1} x_{i_2} \cdots x_{i_s}$, where $\{i_1, i_2, \ldots, i_s\} \subseteq \{1, 2, \ldots, k\}$, $s \geqslant 1$ with the coefficient $r_{i_1} r_{i_2} \cdots r_{i_s}$ except the term $x_1 x_2 \cdots x_k$ for which the coefficient is $r_1 r_2 \cdots r_k - 1$. That is*

$$F(G(n)) = \sum_{\substack{i_1 < i_2 < \cdots < i_s \\ \emptyset \neq \{i_1, i_2, \ldots, i_s\} \subsetneqq \{1,2,\ldots,k\}}} r_{i_1} r_{i_2} \cdots r_{i_s} \; x_{i_1} x_{i_2} \cdots x_{i_s} + (r_1 r_2 \cdots r_k - 1) \; x_1 x_2 \cdots x_k.$$

*Proof.* By Remark 5, $G(m)$ is the mother graph of $G(n)$ and $C(G(n)) = C(G(m))$, where $m = t_1^{r_1} t_2^{r_2} \cdots t_k^{r_k}$, such that $t_i$ is the $i^{\text{th}}$ prime, $i = 1, 2, \ldots, k$. Also the coding of $G(m)$ is unique as it involves all the proper divisors of $m$. Let $\{m_1, m_2, \ldots, m_r\}$ be the set of all proper divisors of $m$, where $r = (r_1 + 1)(r_2 + 1) \cdots (r_k + 1) - 2$. Let us make each $m_i$ square-free and arrange them according to the non-decreasing order to obtain $C(G(m)) = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. Each $\lambda_i$ is a product of primes of the form $t_{i_1} t_{i_2} \cdots t_{i_s}$, $(1 \leqslant s \leqslant k)$ and this particular number repeats, say, $\alpha$ times in the sequence $C(G(m))$, where $\alpha$ is the number of proper divisors of $m$ which are of the form $t_{i_1}^{q_{i_1}} t_{i_2}^{q_{i_2}} \cdots t_{i_s}^{q_{i_s}}$, $1 \leqslant q_{i_j} \leqslant r_{i_j}$ for $j = 1, 2, \ldots, s$, i.e., $\alpha = \prod_{j=1}^{s} r_{i_j}$ for $s < k$ and $\alpha = (\prod_{j=1}^{k} r_j) - 1$ for $s = k$. Thus by definition of $F(G)$ we have

$$F(G(n)) = F(G(m)) = \sum_{\substack{i_1 < i_2 < \cdots < i_s \\ \emptyset \neq \{i_1, i_2, \ldots, i_s\} \subsetneqq \{1,2,\ldots,k\}}} r_{i_1} r_{i_2} \cdots r_{i_s} \; x_{i_1} x_{i_2} \cdots x_{i_s} + (r_1 r_2 \cdots r_k - 1) \; x_1 x_2 \cdots x_k.$$

□

For example, $F(G(p^2 qr)) = F(G(60)) = 2x_1 + x_2 + x_3 + 2x_1 x_2 + 2x_1 x_3 + x_2 x_3 + x_1 x_2 x_3$, where $p, q, r$ are distinct primes.

## Conclusion

There are various representations of simple undirected graphs in terms of adjacency matrices, adjacency lists, unordered pairs etc. But none of them is unique for isomorphic graphs. The importance of the code $C(G)$ is its uniqueness. It is same for any set of isomorphic graphs. The determination of $C(G)$ is not always easy, but once it is obtained for a graph it becomes the characteristic of the graph. Authors believe that the study of the mother graph, the code and the normal polynomial representation of a simple undirected graph will be helpful in further research on graph theory. The purpose of this note is to communicate these interesting observations to all graph theorists.

## References


[1] Ivy Chakrabarty, Shamik Ghosh, T. K. Mukherjee and M. K. Sen, Intersection graphs of ideals of rings, Discrete Math. **309** (2009), 5381–5392.